Mat 241 Chapter 14 Key Test **Name ______________________________________**
Fall 2009, Professor Schultz
Directions: Make sure you show all steps and clearly identify your answers. Good Luck!

#1. (10 points) Find and classify all critical points of the given function.

$f(x, y) = x^4 + y^2 - 8x^2 - 6y + 16$

$f_x(x, y) = 4x^3 - 16x; \quad f_x = 0 \Rightarrow 4x(x+2)(x-2) = 0$

$\therefore x = 0, \pm 2$

$f_y(x, y) = 2y - 6; \quad f_y = 0 \Rightarrow y = 3$

$\therefore y = 3$

$Hence: (0,3), (2,3), and\ (-2,3)$ **are our critical points.**

$f_{xx}(x, y) = 12x^2 - 16; f_{xy}(x, y) = 0; f_{yy}(x, y) = 2$

$D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - \left[f_{xy}(x, y) \right]^2$

$c.p.(0,3) \rightarrow f_{xx}(0,3) = -16 < 0; D(0,3) = -32 < 0$

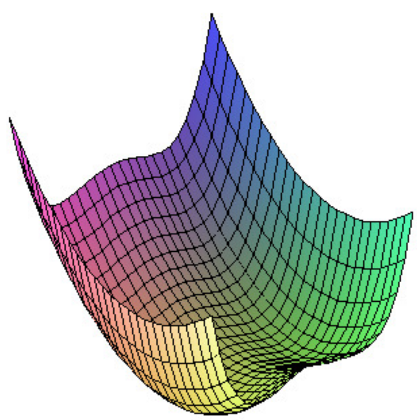
$\therefore (0,3)$ **is a saddle point**

$c.p.(2,3) \rightarrow f_{xx}(2,3) = 32 > 0; D(0,3) = 64 > 0$

$\therefore (2,3)$ **is a relative minimum.**

$c.p.(2,3) \rightarrow f_{xx}(-2,3) = 32 > 0; D(0,3) = 64 > 0$

$\therefore (-2,3)$ **is a relative minimum.**



#2. (5 points) Find the *exact* equation of the tangent plane to the surface $f(x,y)=\sin(xy)$ at the point $\left(\frac{1}{2},\frac{2\pi}{3},\frac{\sqrt{3}}{2}\right)$.

$$f_x(x,y)=y\cos(xy) \qquad f_y(x,y)=x\cos(xy)$$

$$f_x\left(\frac{1}{2},\frac{2\pi}{3}\right)=\frac{2\pi}{3}\cos\left(\frac{\pi}{3}\right)=\frac{\pi}{3} \qquad f_y\left(\frac{1}{2},\frac{2\pi}{3}\right)=\frac{1}{2}\cos\left(\frac{\pi}{3}\right)=\frac{1}{4}$$

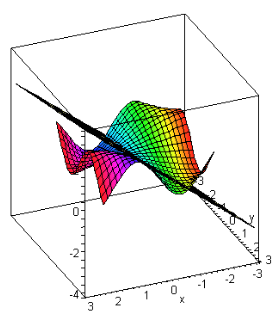
$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

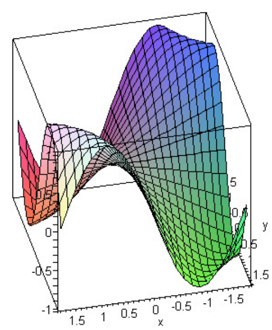
$$z-\frac{\sqrt{3}}{2}=\frac{\pi}{3}\left(x-\frac{1}{2}\right)+\frac{1}{4}\left(y-\frac{2\pi}{3}\right)$$

$$z-\frac{\sqrt{3}}{2}=\frac{\pi x}{3}-\frac{\pi}{6}+\frac{y}{4}-\frac{\pi}{6}$$

$$12z-6\sqrt{3}=4\pi x-4\pi+3y$$

$$\therefore 4\pi x+3y-12z=4\pi-6\sqrt{3}$$



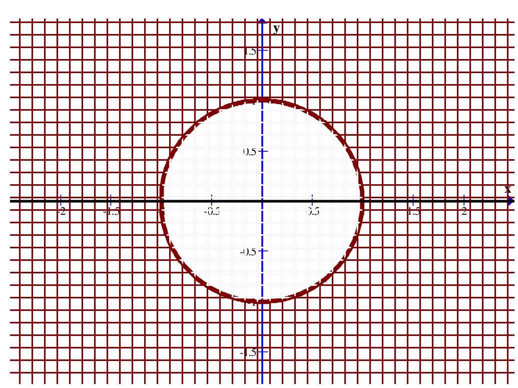


#3. (4 points total)

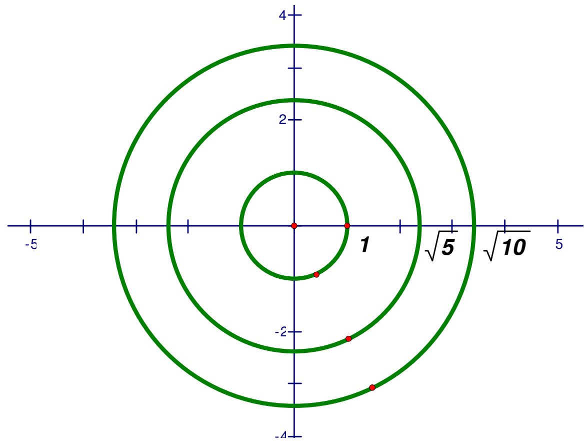
A. Sketch the domain of the following function: $G(x,y)=\sqrt{x^2+y^2-1}$

$$x^2+y^2-1\geq 0 \Leftrightarrow x^2+y^2\geq 1$$

$$D:\{(x,y)\mid x^2+y^2\geq 1\}$$



B. Using the K values 0, 2, and 3, sketch the levels curves.



$$\boldsymbol{K} = 0\text{:}\quad 0 = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2 - 1} \Leftrightarrow \boldsymbol{x}^2 + \boldsymbol{y}^2 = 1 \therefore \boldsymbol{r} = 1$$

$$\boldsymbol{K} = 2\text{:}\quad 2 = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2 - 1} \Leftrightarrow \boldsymbol{x}^2 + \boldsymbol{y}^2 = 5 \therefore \boldsymbol{r} = \sqrt{5}$$

$$\boldsymbol{K} = 3\text{:}\quad 3 = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2 - 1} \Leftrightarrow \boldsymbol{x}^2 + \boldsymbol{y}^2 = 10 \therefore \boldsymbol{r} = \sqrt{10}$$

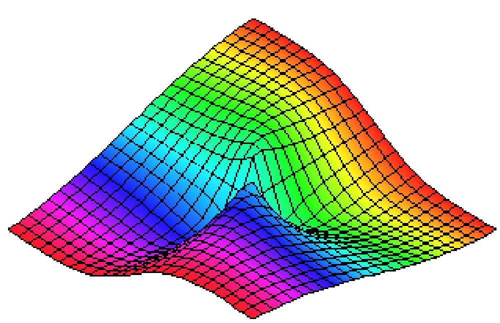
#4. (3 points) Show that the limit doesn't exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$$

$$\textit{approach along } y = x\textbf{,}\textit{then} \quad \lim_{(x,x)\to(0,0)} \frac{x^3 x}{x^6 + x^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^4 + 1} = \frac{0}{1} = 0$$

$$\textit{approach along } y = x^3\textbf{,}\textit{then} \quad \lim_{(x,x)\to(0,0)} \frac{x^3 x^3}{x^6 + \left(x^3\right)^2} = \lim_{(x,x)\to(0,0)} = \frac{1}{2}$$

Since our function approaches two different limits using the paths y = x and y = x^3, we can conclude that the limit does not exist.



#5. (3 points) If $z=\cos(xy)+y\cos(x)$, where $x=u^2+v$ and $y=u-v^2$ **use the chain rule to find**
$\frac{\partial z}{\partial u}$ & $\frac{\partial z}{\partial v}$.

$z=\cos(xy)+y\cos(x)$, where $x=u^2+v$ and $y=u-v^2$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u}=\left(-y\sin(xy)-y\sin(x)\right)2u+\left(-x\sin(xy)+\cos(x)\right)\cdot 1$$

$$\therefore\frac{\partial z}{\partial u}=\cos(x)-x\sin(xy)-2uy\left(\sin(xy)+\sin(x)\right)$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial u}=\left(-y\sin(xy)-y\sin(x)\right)\cdot 1+\left(-x\sin(xy)+\cos(x)\right)\left(-2v\right)$$

$$\therefore\frac{\partial z}{\partial u}=2\left(vx\sin(xy)-v\cos(x)\right)-y\left(\sin(xy)+\sin(x)\right)$$

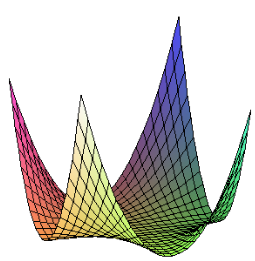
#6. (5 points) Find the directional derivative of $f\left(x,y\right)=\left(xy+1\right)^2$ **at the point (3, 2) in the direction** $\langle 5,3\rangle$.

$$\nabla f\left(x,y\right)=\left\langle 2y\left(xy+1\right),2x\left(xy+1\right)\right\rangle:\nabla f\left(3,2\right)=\left\langle 28,42\right\rangle$$

$$\vec{v}=\langle 5,3\rangle:\vec{u}_{\vec{v}}=\frac{1}{\sqrt{5^2+3^2}}\langle 5,3\rangle=\left\langle\frac{5}{\sqrt{34}},\frac{3}{\sqrt{34}}\right\rangle$$

$$D_{\vec{v}}f\left(3,2\right)=\nabla f\left(3,2\right)\cdot\vec{u}_{\vec{v}}=\langle 28,42\rangle\cdot\left\langle\frac{5}{\sqrt{34}},\frac{3}{\sqrt{34}}\right\rangle=\frac{140}{\sqrt{34}}+\frac{126}{\sqrt{34}}=\frac{266}{\sqrt{34}}$$

$$\therefore D_{\vec{v}}f\left(3,2\right)=\frac{266}{\sqrt{34}}\ \textbf{or}\ \frac{133\sqrt{34}}{17}$$



#7. (6 points) Use Lagrange Multipliers to answer the following.
Suppose the temperature of a metal plate is governed by: $T(x,y)=4x^2-4xy+y^2$ **for any point** (x,y)**. An ant begins walking along the circle:** $x^2+y^2=25$**. What are the highest and lowest temperatures encountered by the ant?**

$\nabla T(x,y)=\langle 8x-4y,-4x+2y\rangle$ **&** $\nabla G(x,y)=\langle 2x,2y\rangle \Rightarrow \lambda\nabla G(x,y)=\langle 2x\lambda,2y\lambda\rangle$**:**

$\nabla T(x,y)=\lambda\nabla G(x,y)\Rightarrow 8x-4y=2x\lambda$ **&** $-4x+2y=2y\lambda$

Solving for $\lambda,\lambda=\dfrac{8x-4y}{2x}=\dfrac{4x-2y}{x}$ */* note* **: if x = 0, then y = 0.**

Substituting **:**

$$-4x+2y=2y\left(\frac{4x-2y}{x}\right)\Leftrightarrow -4x^2+2yx=8yx-4y^2$$

$$\therefore 4x^2-6yx-4y^2=0\rightarrow 2(2x-y)(x+2y)=0\Rightarrow y=2x,x=-2y$$

$$x^2+y^2=x^2+(2x)^2=5x^2=25;x=\pm\sqrt{5}\Rightarrow\left(\sqrt{5},2\sqrt{5}\right),\left(-\sqrt{5},-2\sqrt{5}\right)$$

similarily,

$$x^2+y^2=(-2y)^2+y^2=5y^2=25;y=\pm\sqrt{5}\Rightarrow\left(-2\sqrt{5},\sqrt{5}\right),\left(2\sqrt{5},-\sqrt{5}\right)$$

Testing **:** $\left(\sqrt{5},2\sqrt{5}\right),\left(-\sqrt{5},-2\sqrt{5}\right),\left(-2\sqrt{5},\sqrt{5}\right),\left(2\sqrt{5},-\sqrt{5}\right)$ **in** $T(x,y)$

$$T\left(\sqrt{5},2\sqrt{5}\right)=T\left(-\sqrt{5},-2\sqrt{5}\right)=0$$

$$T\left(-2\sqrt{5},\sqrt{5}\right)=T\left(2\sqrt{5},-\sqrt{5}\right)=125$$

Minimum Temp **:** $0^\circ C$ **and** ***Maximum Temp*** **:** $125^\circ C$