# Mat 241 Homework Set 3 – Due ______________

*Professor David Schultz*

***Directions:*** **Show *all algebraic* steps neatly and concisely using *proper mathematical symbolism*. When graphs and technology are to be implemented, do so appropriately.**

**Mechanics:**

**#1.Determine the limit:**

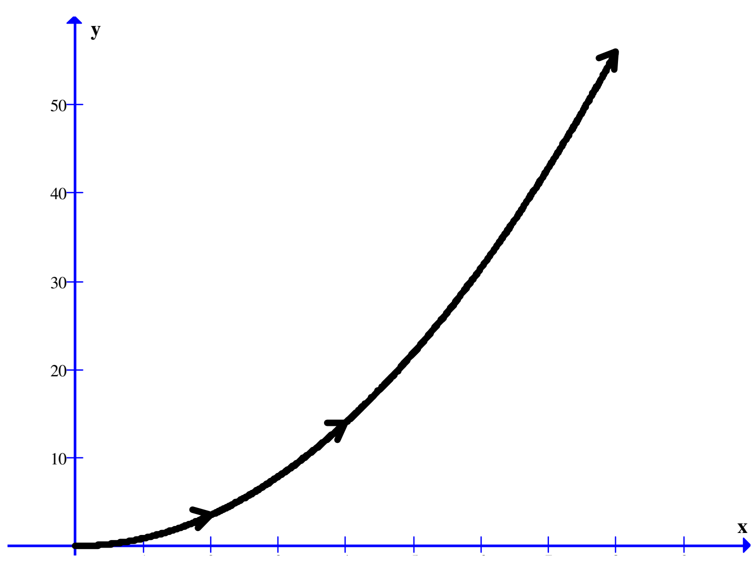
$$\lim_{t\to 1}\left\langle \frac{3}{t^2},\frac{\ln t}{t^2-1},e^t\sin\left(\frac{\pi}{3}t\right)\right\rangle=\left\langle \lim_{t\to 1}\frac{3}{t^2},\lim_{t\to 1}\frac{\ln t}{t^2-1},\lim_{t\to 1}e^t\sin\left(\frac{\pi}{3}t\right)\right\rangle$$

$$=\left\langle 3,\lim_{t\to 1}\frac{\frac{1}{t}}{2t},e\sin\left(\frac{\pi}{3}\right)\right\rangle \text{ /* L'hospital's Rule}$$

$$=\left\langle 3,\frac{1}{2},\frac{e\sqrt{3}}{2}\right\rangle$$

**#2. A particle travels along the path** $\vec{r}(t)=\left\langle 2t,\frac{7}{2}t^2\right\rangle \quad 0\le t\le\infty$ **seconds.**

A. **On a separate graph hand sketch a trace of the vector function with t = 0, 1, 2, and 4 seconds.**

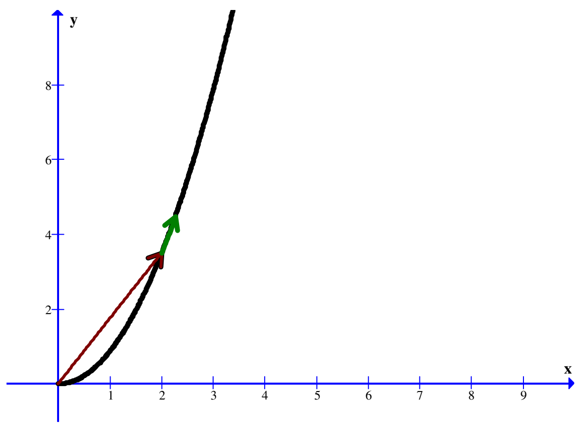


**B. Determine the position vector and the unit tangent vector $\vec{\mathbf{T}}(\mathbf{t})=\dfrac{\vec{\mathbf{r}}'(\mathbf{t})}{\left\|\vec{\mathbf{r}}'(\mathbf{t})\right\|}$ at t = 1 second and hand draw both of these vectors on the graph in part B.**

$$\vec{\mathbf{r}}'(\mathbf{t})=\langle 2,7t\rangle;\left|\vec{\mathbf{r}}'(\mathbf{t})\right|=\left|\sqrt{2^2+(7t)^2}\right|=\sqrt{4+49t^2}$$

$$\vec{\mathbf{r}}'(\mathbf{1})=\langle 2,7\rangle;\left|\vec{\mathbf{r}}'(\mathbf{1})\right|=\sqrt{4+49}=\sqrt{53}$$

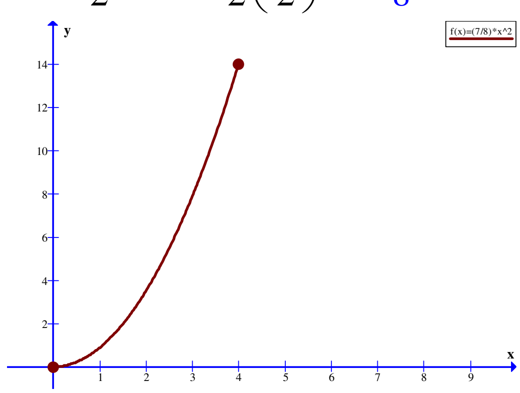
$$\vec{\mathbf{T}}(\mathbf{t})=\frac{\vec{\mathbf{r}}'(\mathbf{t})}{\left\|\vec{\mathbf{r}}'(\mathbf{t})\right\|}\rightarrow\vec{\mathbf{T}}(\mathbf{1})=\frac{\vec{\mathbf{r}}'(\mathbf{1})}{\left\|\vec{\mathbf{r}}'(\mathbf{1})\right\|}=\left\langle\frac{2}{\sqrt{53}},\frac{7}{\sqrt{53}}\right\rangle$$




**C. Eliminate the parameter, t, obtaining a function of the form y=f(x) and hand graph this function over the domain [0,4]. Is this the same graph as in part A?**

$$\vec{r}(t)=\left\langle 2t,\frac{7}{2}t^2\right\rangle\quad\rightarrow x=2t;y=\frac{7}{2}t^2$$

$$so\ t=\frac{x}{2}\Rightarrow y=\frac{7}{2}\left(\frac{x}{2}\right)^2=\frac{7x^2}{8}$$




**Same curve but the vector function includes the direction of trace.**

**#3. Suppose that the velocity of a particle is modeled by the vector function $\vec{v}(t)=\left\langle \dfrac{2t^2}{16-3t^2}, te^{3t}, \sin^3(4t)\right\rangle$ . Find the position vector function $\vec{r}(t)$ and the acceleration vector function $\vec{a}(t)$.**

$$\vec{v}(t)=\left\langle \frac{2t^2}{16-3t^2}, te^{3t}, \sin^3(4t)\right\rangle; \vec{r}(t)=\int \vec{v}(t)dt=\left(\int \frac{2t^2}{16-3t^2}dt\right)\vec{i}+\left(\int te^{3t}dt\right)\vec{j}+\left(\int \sin^3(4t)\right)\vec{k}$$

$\rightarrow$ **looking at each component seperately:**

$$\int \frac{2t^2}{16-3t^2}dt=\int\left(\frac{-2}{3}+\frac{32}{3\left(16-3t^2\right)}\right)dt \text{ /* long division}$$

$$=\frac{-2}{3}\int dt+\frac{32}{3}\int\frac{dt}{16-3t^2}$$

$$=\frac{-2t}{3}+\frac{32}{3}\int\frac{dt}{\left(4+\sqrt{3}t\right)\left(4-\sqrt{3}t\right)} \text{ /* difference of two squares}$$

$$\Rightarrow \frac{1}{\left(4+\sqrt{3}t\right)\left(4-\sqrt{3}t\right)}=\frac{A}{4+\sqrt{3}t}+\frac{B}{4-\sqrt{3}t}$$

$$\Rightarrow 1=4A+4B-A\sqrt{3}t+B\sqrt{3}t=4(A+B)+\left(B\sqrt{3}-A\sqrt{3}\right)t$$

$$1=4(A+B) \text{ \& } 0=\left(B\sqrt{3}-A\sqrt{3}\right)\Leftrightarrow A=B$$

$$A=B=\frac{1}{8}$$

$$=\frac{-2t}{3}+\frac{32}{3}\int\frac{dt}{\left(4+\sqrt{3}t\right)\left(4-\sqrt{3}t\right)}$$

$$=\frac{-2t}{3}+\frac{32}{3}\bullet\frac{1}{8}\int\left(\frac{1}{4+\sqrt{3}t}+\frac{1}{4-\sqrt{3}t}\right)dt$$

$$=\frac{-2t}{3}+\frac{4}{3}\int\frac{dt}{4+\sqrt{3}t}+\frac{4}{3}\int\frac{dt}{4-\sqrt{3}t}$$

$$=\frac{-2t}{3}+\frac{4}{3\sqrt{3}}\ln\left|4+\sqrt{3}t\right|-\frac{4}{3\sqrt{3}}\ln\left|4-\sqrt{3}t\right|$$

$$=\frac{-2t}{3}+\frac{4}{3\sqrt{3}}\ln\left|\frac{4+\sqrt{3}t}{4-\sqrt{3}t}\right|+\vec{c}_1$$

$$\int te^{3t}dt \rightarrow \begin{array}{cc} d & \int \\ t & e^{3t} \\ 1 & \dfrac{e^{3t}}{3} \\ 0 & \dfrac{e^{3t}}{9} \end{array} = \frac{e^{3t}t}{3} - \frac{e^{3t}}{9} + \vec{c}_2$$

$$\int \sin^3(4t)dt = \int \sin(4t)\sin^2(4t)dt = \int \sin(4t)\left(1-\cos^2(4t)\right)dt = \int \sin t dt - \int \sin t \cos^2 t dt$$

$$= \frac{-\cos t}{4} + \frac{1}{4}\int u^2 du \quad /\!*\ u = \cos t$$

$$= \frac{-\cos t}{4} + \frac{\cos^3 t}{12} + \vec{c}_3$$

$$= \frac{\cos^3 t}{12} - \frac{\cos t}{4} + \vec{c}_3$$

$$\vec{r}(t) = \left\langle \frac{4\sqrt{3}}{9}\ln\left|\frac{4+\sqrt{3}t}{4-\sqrt{3}t}\right| - \frac{2t}{3}, \frac{e^{3t}t}{3} - \frac{e^{3t}}{9}, \frac{\cos^3(4t)}{12} - \frac{\cos(4t)}{4} \right\rangle + \vec{C}$$

$$\vec{a}(t) = \vec{v}'(t) = \left\langle \frac{(16-3t^2)4t - 2t^2(-6t)}{(16-3t^2)^2}, 3te^{3t} + e^{3t}, 3\sin^2(4t)4\cos(4t) \right\rangle$$

$$= \left\langle \frac{64t}{(16-3t^2)^2}, 3te^{3t} + e^{3t}, 12\sin^2(4t)\cos(4t) \right\rangle$$

**#4. Suppose the trajectories of two objects are modeled by:**

$$\vec{r}_1(t)=\langle t^2, 7t-12, t^2\rangle;\ \vec{r}_2(t)=\langle 4t-3, t^2, 5t-6\rangle\quad t\geq 0$$

**A. Determine if these particles ever collide.**

$\vec{r}_1(t)=\langle t^2, 7t-12, t^2\rangle;\ \vec{r}_2(t)=\langle 4t-3, t^2, 5t-6\rangle\quad t\geq 0$

$t^2=4t-3 \Leftrightarrow t^2-4t+3=(t-3)(t-1)=0 \therefore t=1,3$

***test*** $t=1$ **in the second component.**

$7\cdot 1-12\neq 1^2 \therefore$ **they do not collide at 1 second**

***test*** $t=3$ **in the second component & third components**

$7\cdot 3-12=3^2 \quad 3^2=5\cdot 3-6$

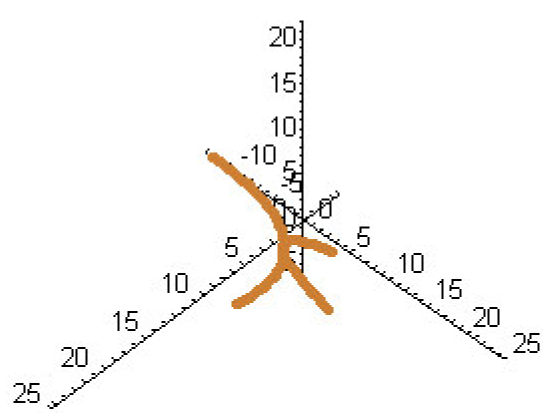
$\therefore$ **They collide at 3 seconds.**

**B. If their paths do cross, how many times does it occur and when?**

**Let t & s be potential times. Then the three components yield:**

$$t^2=4s-3$$
$$7t-12=s^2$$
$$t^2=5s-6$$
$$\textit{equating } 1 \ \& \ 3 \quad 5s-6=4s-3 \rightarrow s=3$$
$$\textit{which implies } t=3.$$

**They only cross paths once, when t is 3 seconds which also happens to be a collision point!**



**#5. Determine the *tangent line* to $\vec{r}(t) = \left\langle \sin\left(\pi e^t\right), \cos\left(\pi e^t\right), e^t \right\rangle$ at t = 0.**

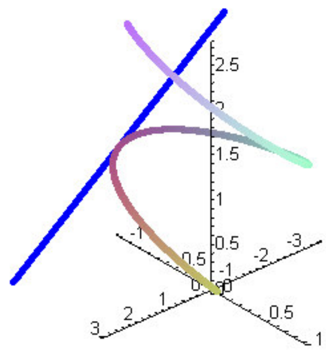
$$\vec{r}(t) = \left\langle \sin\left(\pi e^t\right), \cos\left(\pi e^t\right), e^t \right\rangle; \vec{r}(0) = \langle 0,-1,1\rangle; P_0(0,-1,1)$$

$$\vec{r}'(t) = \left\langle \pi e^t \cos\left(\pi e^t\right), -\pi e^t \sin\left(\pi e^t\right), e^t \right\rangle$$

$$\vec{r}'(0) = \left\langle \pi e^0 \cos\left(\pi e^0\right), -\pi e^0 \sin\left(\pi e^0\right), e^0 \right\rangle = \langle -\pi, 0, 1\rangle$$

$$x = 0 + (-\pi)t, y = -1 + 0t, z = 1 + 1t$$

$$x = -\pi t, y = -1, z = 1 + 1t$$




**<u>Concept Development & Graphics:</u>**

**#6. Compute the tangent vectors and *unit* tangent vectors to the curves:**

$$\vec{r}(t) = \left\langle \cos t, \sin t, \frac{\sqrt{5}}{3} \right\rangle \ \ \& \ \ \vec{s}(t) = \left\langle \cos t, \sin t, -\frac{\sqrt{5}}{3} \right\rangle$$

**A. Algebraically show that these two vectors form the intersection of the ellipsoid $\frac{4}{9}x^2 + \frac{4}{9}y^2 + z^2 = 1$ and the cylinder $x^2 + y^2 = 1$.**

$$\vec{r}(t) = \left\langle \cos t, \sin t, \frac{\sqrt{5}}{3} \right\rangle \ \ \& \ \ \vec{s}(t) = \left\langle \cos t, \sin t, -\frac{\sqrt{5}}{3} \right\rangle$$

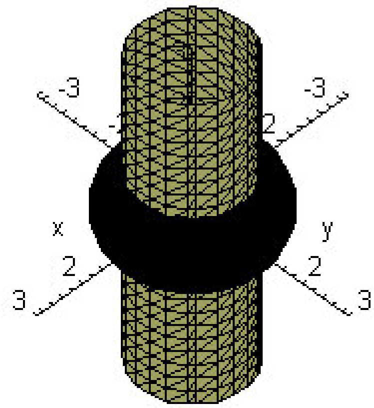
$$x = \cos t, y = \sin t, z = \frac{\sqrt{5}}{3} \quad \& \quad x = \cos t, y = \sin t, z = -\frac{\sqrt{5}}{3}$$

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1; z^2 = \frac{5}{9} \text{ for both}$$

$$\frac{4}{9}x^2 + \frac{4}{9}y^2 + z^2 = \frac{4}{9}\left(x^2 + y^2\right) + z^2 = \frac{4}{9}\cdot 1 + \frac{5}{9} = 1$$

$$\therefore \textit{they lie on the ellipsoid. The cylinder is trivial.}$$

B. Use Maple or some other software package to graph the two surfaces on a grid (see helpful code). Make sure they are both on the same graph.




C. Compute the tangent vectors and *unit* tangent vectors to the curves. See page 858 example 1. Why do these two vectors have zero $\vec{k}$ components?

$$\vec{r}(t)=\left\langle \cos t,\sin t,\frac{\sqrt{5}}{3}\right\rangle \;\&\; \vec{s}(t)=\left\langle \cos t,\sin t,-\frac{\sqrt{5}}{3}\right\rangle$$

$$\vec{r}'(t)=\langle -\sin t,\cos t,0\rangle \;\&\; \vec{s}'(t)=\langle -\sin t,\cos t,0\rangle$$

$$\vec{T}_{\vec{r}(t)}=\langle -\sin t,\cos t,0\rangle \;\&\; \vec{T}_{\vec{s}(t)}=\langle -\sin t,\cos t,0\rangle$$

Their k components were constants.

#7. I claim that $T'(t)\perp T(t)$. Verify this result for the case when

$$\vec{r}(t)=t^3\vec{i}+t^6\vec{j}$$

***I will use some space saving variable assignment techniques which you should try to utilize in your future efforts.***

$$\vec{r}(t) = t^3\vec{i} + t^6\vec{j} = \langle t^3, t^6 \rangle$$

$$\vec{r}'(t) = \langle 3t^2, 6t^5 \rangle = (3t^2)\langle 1, 2t^3 \rangle; |\vec{r}'(t)| = \sqrt{9t^4 + 36t^{10}} = 3t^2\sqrt{1+4t^6}$$

$$\textit{Let} \ \ u = \sqrt{1+4t^6} \ \ \textit{then} \ \ u^2 = 1 + 4t^6 \ \ \textbf{and} \ \ u' = \frac{24t^5}{2\sqrt{1+4t^6}} = \frac{12t^5}{u}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{(3t^2)\langle 1, 2t^3 \rangle}{3t^2\sqrt{1+4t^6}} = \left\langle \frac{1}{u}, \frac{2t^3}{u} \right\rangle \ \textbf{and} \ \vec{T}'(t) = \left\langle \frac{-u'}{u^2}, \frac{6t^2u - 2t^3u'}{u^2} \right\rangle$$

$$\vec{T}'(t) \cdot \vec{T}(t) = \left\langle \frac{-u'}{u^2}, \frac{6t^2u - 2t^3u'}{u^2} \right\rangle \cdot \left\langle \frac{1}{u}, \frac{2t^3}{u} \right\rangle$$

$$= \frac{-u'}{u^3} + \frac{2t^3(6t^2u - 2t^3u')}{u^3}$$

$$= \frac{-\frac{12t^5}{u} + 12t^5u - 4t^6\left(\frac{12t^5}{u}\right)}{u^3}$$

$$= \frac{-12t^5 + 12t^5(1+4t^6) - 48t^{11}}{u^4}$$

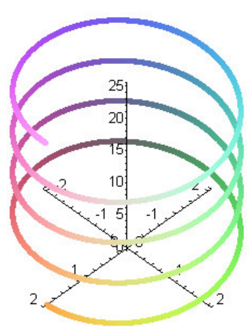
$$= \frac{-12t^5 + 12t^5 + 48t^{11} - 48t^{11}}{(1+4t^6)^2}$$

$$= 0$$

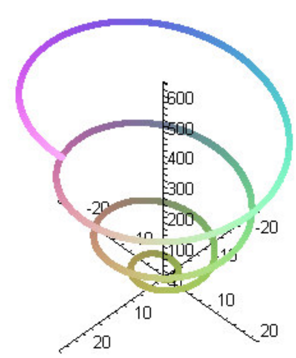
**#8. There are several space curves that are quite interesting. Use a plotting software package like Maple to plot the 4 curves listed below.**

**A. & B. Elliptical & Conical Helices:**

$2\cos t\vec{i} + 2\sin t\vec{j} + t\vec{k}$

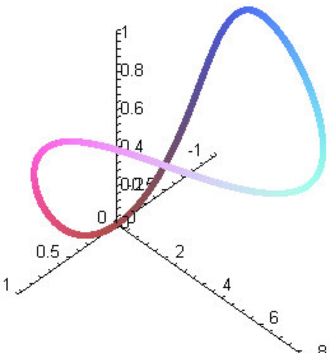


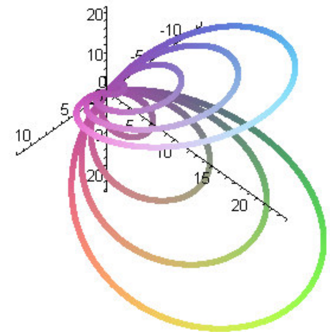

$\vec{r}(t) = \langle t\cos t, t\sin t, t^2 \rangle$



**B. & D Other spacecurves.**

$\vec{r}(t) = \langle \sin t, (1-\cos t)^3, \sin(1-\cos t) \rangle$ $\qquad$ $\vec{r}(t) = \langle t \sin t \cos t, t \sin^2 t, t \sin t \rangle$



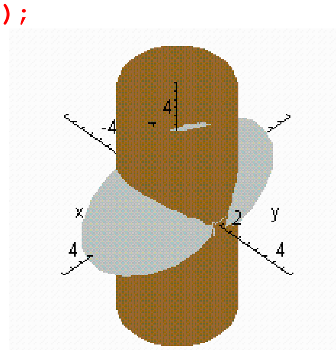




**To plot surfaces use this Maple code: Example:**

**A. Graph the following two surfaces: The ellipsoid:** $\dfrac{x^2}{16} + \dfrac{y^2}{4} + \dfrac{z^2}{4} = 1$

**and the cylinder:** $x^2 + y^2 = 4$

```
> restart:with(plots):with(plottools):
Warning, the name changecoords has been redefined
Warning, the assigned name arrow now has a global binding
> a:=implicitplot3d(x^2/(16)+y^2/4+z^2/4=1, x=-5..5,y=-5..5,z=-
5..5,numpoints=10000,color=gray,style = patchnogrid):
> b:=implicitplot3d(x^2+y^2=4, x=-5..5,y=-5..5,z=-
5..5,numpoints=10000,color=sienna,style=patchnogrid):
> display(a,b,axes=normal);
```



**B. Determine parametric equations for the space curve of intersection. Then plot the surfaces and the space curve of intersection on the same graph**

**Let** $x = 2\cos(t)$ **and** $y = 2\sin(t) \rightarrow \textbf{\textit{note}}: x^2 + y^2 = 4$

**Using the above choice for x & y, find the z(t) function(s) from the ellipsoid.**

$$\frac{x^2}{16}+\frac{y^2}{4}+\frac{z^2}{4}=1 \Leftrightarrow \frac{4\cos^2 t}{16}+\frac{4\sin^2 t}{4}+\frac{z^2}{4}=1 \Leftrightarrow z^2 = 4-\cos^2 t - 4\sin^2 t$$

$$\therefore z(t)=\sqrt{4-\cos^2 t-4\sin^2 t} \text{ and } z(t) = -\sqrt{4-\cos^2 t-4\sin^2 t}$$

```
> c:=spacecurve([2*cos(t),2*sin(t), sqrt(4-(cos(t))^2-
4*(sin(t))^2)],t=0..2*Pi,thickness=3,color=black):
d:=spacecurve([2*cos(t),2*sin(t), -sqrt(4-(cos(t))^2-
4*(sin(t))^2)],t=0..2*Pi,thickness=3,color=black):
display(a,b,c,d,axes=normal);
```