Mat 241 Homework Set 4 Key – Due ______________

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Directions: **Show *all algebraic* steps neatly and concisely using *proper mathematical symbolism*. When graphs and technology are to be implemented, do so appropriately.**

Mechanics:

#1. We discussed how the parametrization of a curve is not unique.
Consider the vector function: $\vec{r}(t) = \sin t\vec{i} + t\vec{j} + \cos t\vec{k} \quad 0 \le t \le 2\pi$
Compute the arc length using the various parametrizations:

A.

$$L = \int_0^{2\pi} |\vec{r}'(t)| dt$$

$$\vec{r}'(t) = \langle \cos t, 1, -\sin t \rangle; |\vec{r}'(t)| = \sqrt{\cos^2 t + 1 + \sin^2 t} = \sqrt{2}.$$

$$L = \int_0^{2\pi} |\vec{r}'(t)| dt = \int_0^{2\pi} \sqrt{2} dt = \sqrt{2}t \Big|_0^{2\pi} = 2\sqrt{2}\pi$$

B.

$$L = \int_0^{\sqrt{2\pi}} |\vec{s}'(t)| dt, \text{ where } \vec{s}(t) = \langle \sin t^2, t^2, \cos t^2 \rangle, 0 \le t \le \sqrt{2\pi}$$

$$\vec{s}'(t) = \langle 2t\cos t^2, 2t, -2t\sin t^2 \rangle; |\vec{s}'(t)| = \sqrt{4t^2\cos^2 t^2 + 4t^2 + 4t^2\sin^2 t^2} = 2\sqrt{2}t$$

$$L = \int_0^{\sqrt{2\pi}} |\vec{s}'(t)| dt = \int_0^{\sqrt{2\pi}} 2\sqrt{2}t dt = \sqrt{2}t^2 \Big|_0^{\sqrt{2\pi}} = 2\sqrt{2}\pi$$

C.

$$L = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} |\vec{v}'(s)| ds, \text{ where } \vec{v}(s) = \left\langle \cos(s), s + \frac{\pi}{2}, -\sin(s) \right\rangle, -\frac{\pi}{2} \le s \le \frac{3\pi}{2}$$

$$\vec{v}'(s) = \langle -\sin(s), 1, -\cos(s) \rangle; |\vec{v}'(s)| = \sqrt{\sin^2(s) + 1 + \cos^2(s)} = \sqrt{2}$$

$$L = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} |\vec{v}'(s)| ds = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{2} ds = \sqrt{2}t \Big|_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{3\sqrt{2}\pi}{2} + \frac{\sqrt{2}\pi}{2} = \frac{4\sqrt{2}\pi}{2} = 2\sqrt{2}\pi$$

#2. Given the position vector $\vec{r}(t)=\left\langle \frac{t^2}{2},\frac{t^3}{3},t \right\rangle$ **meters**

A. Determine the velocity vector $\vec{v}(t)$ **and the acceleration vector** $\vec{a}(t)$**.**

$$\vec{v}(t)=\langle t,t^2,1\rangle; \quad \vec{a}(t)=\vec{v}'(t)=\langle 1,2t,0\rangle$$

B. Determine the speed, curvature, and the tangential (a_t) and the normal (a_N) of the acceleration.

$$\textit{speed}: |\vec{v}(t)|=\sqrt{t^2+t^4+1}=\sqrt{t^4+t^2+1}$$

$$\vec{r}'(t)=\vec{v}(t)=\langle t,t^2,1\rangle; \vec{r}''(t)=\vec{a}(t)=\langle 1,2t,0\rangle$$

$$\vec{r}'(t)\times\vec{r}''(t)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ t & t^2 & 1\\ 1 & 2t & 0\end{vmatrix}=(0-2t)\vec{i}-(0-1)\vec{j}+\left(2t^2-t^2\right)\vec{k}=\langle -2t,1,t^2\rangle$$

$$|\vec{r}'(t)\times\vec{r}''(t)|=\sqrt{4t^2+1+t^4}=\sqrt{t^4+4t^2+1}$$

$$|\vec{r}'(t)|=\sqrt{t^4+t^2+1}\rightarrow|\vec{r}'(t)|^3=\left(t^4+t^2+1\right)\sqrt{t^4+t^2+1}$$

$$\textit{curvature}: \ \kappa=\frac{\sqrt{t^4+4t^2+1}}{\left(t^4+t^2+1\right)\sqrt{t^4+t^2+1}}$$

$$a_T=v'=\frac{\vec{v}\cdot\vec{a}}{v}=\frac{\vec{r}'(t)\cdot\vec{r}''(t)}{|\vec{r}'(t)|}\frac{\langle t,t^2,1\rangle\cdot\langle 1,2t,0\rangle}{\sqrt{t^4+t^2+1}}=\frac{2t^3+t}{\sqrt{t^4+t^2+1}}$$

$$a_N=\kappa v^2=\frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|}=\frac{\sqrt{t^4+4t^2+1}}{\sqrt{t^4+t^2+1}}$$

C. **Determine the unit vectors: $\vec{T}, \vec{N}$, and $\vec{B}$ at t = 1 second.**

$$\vec{T}(1) = \frac{\vec{r}'(1)}{|\vec{r}'(1)|} = \frac{\vec{v}(1)}{v(1)} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$v'(t) = \frac{4t^3 + 2t}{2\sqrt{t^4 + t^2 + 1}} = \frac{2t^3 + t}{\sqrt{t^4 + t^2 + 1}} : v'(1) = \frac{3}{\sqrt{3}}$$

$$\vec{T}'(1) = \frac{v(1)\vec{v}'(1) - \vec{v}(1)v'(1)}{[v(1)]^2} = \frac{\sqrt{3}\langle 1,2,0\rangle - \langle 1,1,1\rangle \frac{3}{\sqrt{3}}}{[\sqrt{3}]^2} = \left\langle 0, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle$$

$$|\vec{T}'(1)| = \sqrt{0^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\vec{N}(1) = \frac{\vec{T}'(1)}{|\vec{T}'(1)|} = \frac{\sqrt{3}}{\sqrt{2}}\left\langle 0, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle = \left\langle 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$$

$$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} = \left(-\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}\right)\vec{i} - \left(-\frac{1}{\sqrt{6}} - 0\right)\vec{j} + \left(\frac{1}{\sqrt{6}} - 0\right)\vec{k}$$

$$= \left\langle -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

D. **Express the acceleration in terms of the moving trihedral ($\vec{T}, \vec{N}$, and $\vec{B}$).**

$$\vec{a}(t) = a_T(t)\vec{T}(t) + a_N(t)\vec{N}(t)$$

$$\therefore \vec{a}(1) = a_T(1)\vec{T}(1) + a_N(1)\vec{N}(1)$$

$$a_T(1) = \frac{3}{\sqrt{3}}; a_N(1) = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$$

$$\therefore \vec{a}(1) = \frac{3}{\sqrt{3}}\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle + \sqrt{2}\left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = \langle 1,1,1\rangle + \langle 0,1,-1\rangle = \langle 1,2,0\rangle$$

Note! Let t = 1 in the acceleration vector from part A. Does this check? ☺

#3. If a particle with mass m moves with position vector $\vec{r}(t)$, then its *angular momentum* is defined to be $\vec{L}(t)=m\vec{r}(t)\times\vec{v}(t)$ and its torque as $\vec{\tau}(t)=m\vec{r}(t)\times\vec{a}(t)$. Show that $\vec{L}'(t)=\vec{\tau}(t)$. Note: if the torque vector $\vec{\tau}(t)$ is the zero vector, then the angular momentum is constant (i.e. $\vec{L}(t)=k$. This is the *law of conservation of momentum*.

$$\begin{aligned}
\vec{L}(t)&=m\vec{r}(t)\times\vec{v}(t)\\
\vec{L}'(t)&=m\vec{r}'(t)\times\vec{v}(t)+m\vec{r}(t)\times\vec{v}'(t)\\
&=m\vec{v}(t)\times\vec{v}(t)+m\vec{r}(t)\times\vec{a}(t)\\
&=m\vec{0}+m\vec{r}(t)\times\vec{a}(t)\\
&=\vec{0}+m\vec{r}(t)\times\vec{a}\\
&=m\vec{r}(t)\times\vec{a}=\vec{\tau}(t)
\end{aligned}$$

#4. The torsion is a scalar defined to be $\tau(t)=\dfrac{\left(\vec{r}'(t)\times\vec{r}''(t)\right)\cdot\vec{r}'''(t)}{\left|\vec{r}'(t)\times\vec{r}''(t)\right|}$.

Find the torsion of the curve $x=\sinh t, y=\cosh t, z=t$ at the point (0, 1, 0). Page 252 may help.

The point (0, 1, 0) implies that t = 0.

$$\vec{r}'(t)=\langle\cosh t,\sinh t,1\rangle;\vec{r}'(0)=\langle 1,0,1\rangle$$

$$\vec{r}''(t)=\langle\sinh t,\cosh t,0\rangle;\vec{r}''(0)=\langle 0,1,0\rangle$$

$$\vec{r}'''(t)=\langle\cosh t,\sinh t,0\rangle;\vec{r}'''(0)=\langle 1,0,0\rangle$$

$$\vec{r}'(0)\times\vec{r}''(0)=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&0&1\\0&1&0\end{vmatrix}=(0-1)\vec{i}-(0-0)\vec{j}+(1-0)\vec{k}=\langle -1,0,1\rangle$$

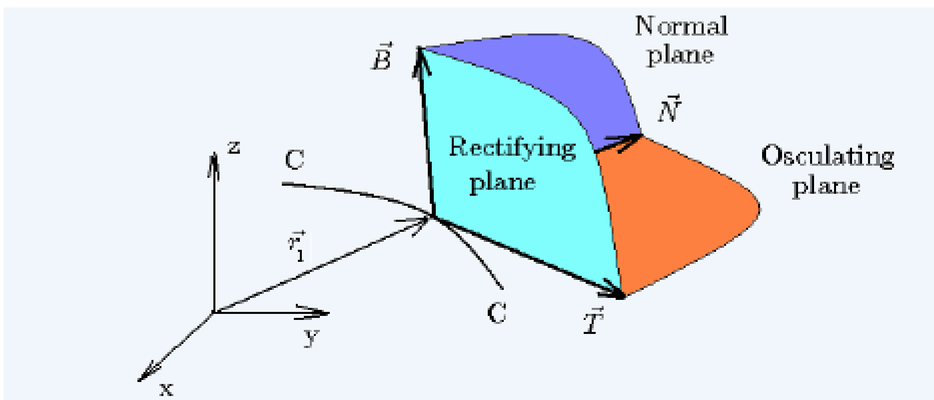
$$\left|\vec{r}'(0)\times\vec{r}''(0)\right|=\sqrt{1+0+1}=\sqrt{2}$$

$$\left(\vec{r}'(0)\times\vec{r}''(0)\right)\cdot\vec{r}'''(t)=\langle -1,0,1\rangle\cdot\langle 1,0,0\rangle=-1$$

$$\tau(0)=\frac{\left(\vec{r}'(0)\times\vec{r}''(0)\right)\cdot\vec{r}'''(0)}{\left|\vec{r}'(0)\times\vec{r}''(0)\right|}=\frac{-1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$$

Why Is the Moving Triad Important?

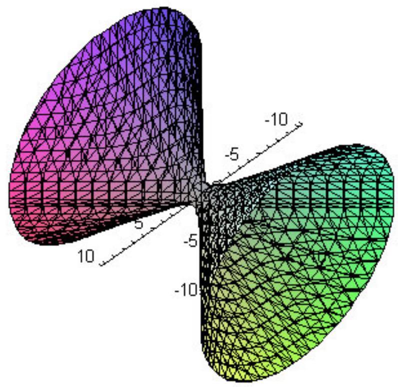
The moving triad is not only a mathematical concept. It also provides vital information of the characteristics of a moving object. For example, if you are on an airplane or a roller-coaster, as you are "flying" you are following a curve. Therefore, you are moving in the direction of the tangent vector, your "up" vector is in the direction of the binormal vector and the rate of turning and turning direction are given by the curvature and the direction of the normal vector, respectively. That is, while you are rolling or tumbling, this triad always provides you with the forward, up and turning directions. Therefore, to correctly animate a moving object, you need to know the geometric characteristics of the moving triad. There is one more parameter, the *torsion*, that describing the way of twisting of a space curve.



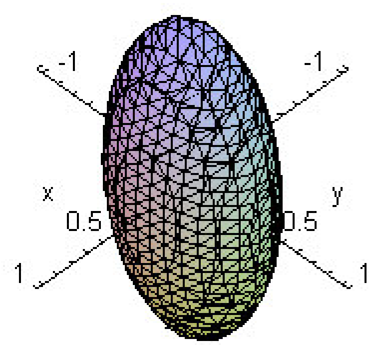
Note: This is often called the Frenet frame or TNB-frame.

Identify the following 4 surfaces and sketch them by hand on graph paper noting the important points and traces. Note: you may have to put them in standard form.

A. $x^2 - y^2 + z^2 = 1$ **Hyperboloid of one sheet with y as the axis of symmetry.**

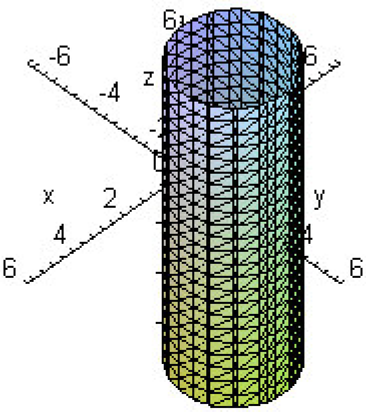


B. $9x^2 + 4y^2 + z^2 = 1 \Leftrightarrow \dfrac{x^2}{1/9} + \dfrac{y^2}{1/4} + z^2 = 1$



C. $x^2 + y^2 = 4y \Leftrightarrow x^2 + y^2 - 4y = 0 \Leftrightarrow x^2 + (y-2)^2 = 4$

Cylinder or radius 2 with xy-center (0,2).



D. $x^2 + 2z^2 = 1 \Leftrightarrow \dfrac{x^2}{1} + \dfrac{z^2}{\frac{1}{2}} = 1$ **An ellipse shaped surface with y as the axis of symmetry.**