

Mat 241 Homework Set 8 – Due _____

Professor David Schultz

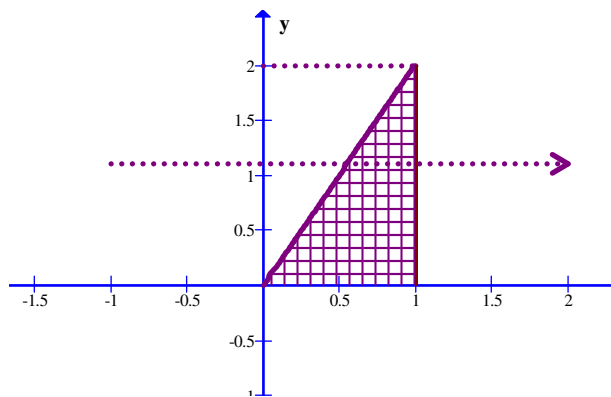
Directions: Show all algebraic steps neatly and concisely using proper mathematical symbolism. When graphs and technology are to be implemented, do so appropriately.

Mechanics:

For the following two integrals do the following:

- Sketch the region of integration for the two integrals shown.
- Compute the integrals in 1 & 2 exactly.
- Write each integral with the order of integration reversed and then compute each of the “new” integrals.

1.



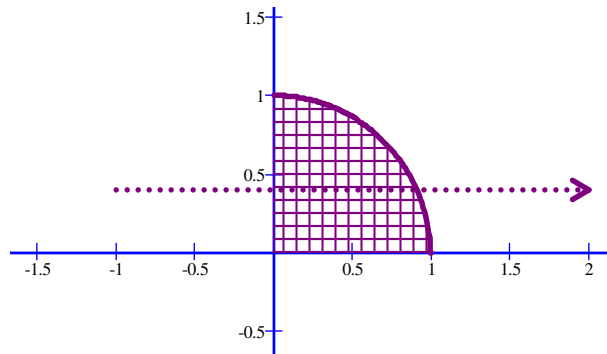
$$\int_0^2 \int_{\frac{y}{2}}^1 (x + y) dx dy$$

$$= \int_0^2 \left(\frac{x^2}{2} + yx \Big|_{\frac{y}{2}}^1 \right) dy = \int_0^2 \left(\frac{1}{2} + y - \frac{5y^2}{8} \right) dy = \left(\frac{y}{2} + \frac{y^2}{2} - \frac{5y^3}{24} \right) \Big|_0^2 = \frac{4}{3}$$

$$\int_0^2 \int_{\frac{y}{2}}^1 (x + y) dx dy = \int_0^1 \int_0^{2x} (x + y) dy dx$$

$$= \int_0^1 \left(xy + \frac{y^2}{2} \Big|_0^{2x} \right) dx = \int_0^1 (4x^2) dx = \left(\frac{4x^3}{3} \right) \Big|_0^1 = \frac{4}{3}$$

2.



$$\int_0^1 \int_0^{\sqrt{1-y^2}} y dx dy$$

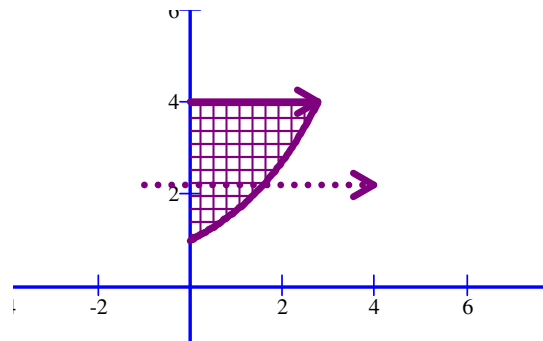
$$= \int_0^1 yx \Big|_0^{\sqrt{1-y^2}} dy = \int_0^1 y\sqrt{1-y^2} dy = \frac{1}{2} \int_0^1 \sqrt{u} du = \frac{1}{2} \frac{2}{3} u^{\frac{3}{2}} \Big|_0^1 = \frac{1}{3}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} y dy dx = \int_0^1 \frac{y^2}{2} \Big|_0^{\sqrt{1-x^2}} dx = \frac{1}{2} \int_0^1 (1-x^2) dx = \left(x - \frac{x^3}{3} \right) \Big|_0^1 = \frac{1}{3}$$

For the following two integrals do the following:

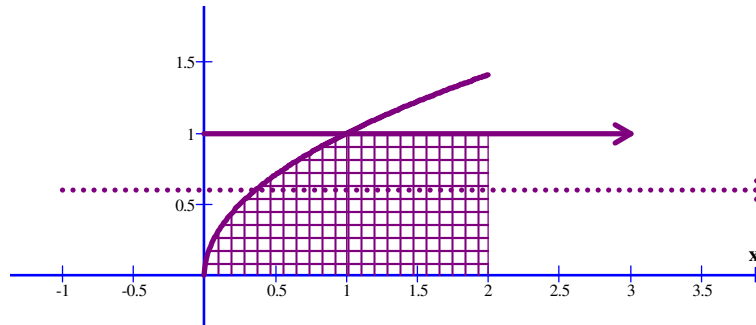
- D. Sketch the region of integration.
- E. Write each integral with the order of integration reversed.
- F. Compute each of the “new” integrals. Hint: On #4 utilize a Taylor series using the first 6 terms to approximate one part of the region.

$$3. \int_0^{\ln 16} \int_{\frac{x}{e^2}}^4 \frac{1}{\ln y} dy dx = \int_1^4 \int_0^{2 \ln y} \frac{1}{\ln y} dx dy = \int_1^4 \frac{x}{\ln y} \Big|_0^{2 \ln y} dy = \int_1^4 2 dy = 6$$



4.

$$\begin{aligned}
 \int_0^1 \int_{y^2}^2 \frac{e^x}{\sqrt{x}} dx dy &= \int_0^1 \int_0^{\sqrt{x}} \frac{e^x}{\sqrt{x}} dy dx + \int_1^2 \int_0^1 \frac{e^x}{\sqrt{x}} dy dx = \int_0^1 \frac{e^x}{\sqrt{x}} \Big|_0^{\sqrt{x}} dx + \int_1^2 \frac{e^x}{\sqrt{x}} \Big|_0^1 dx \\
 &= \int_0^1 e^x dx + \int_1^2 \frac{e^x}{\sqrt{x}} dx = e - 1 + \int_1^2 \left(\frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}} + \frac{x^2}{2\sqrt{x}} + \dots \right) dx \\
 &\approx e - 1 + \int_1^2 \left(\frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}} + \frac{x^2}{2\sqrt{x}} + \frac{x^3}{6\sqrt{x}} \right) dx \\
 &\rightarrow e - 1 + 3.470 \\
 &\therefore e + 2.47
 \end{aligned}$$



Concept development and applications.

#5. Use a CAS to compute the following two integrals showing that they are not the same. Why doesn't this contradict Fubini's Theorem?

$$\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dy dx \quad \& \quad \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dx dy$$

The function has an infinite discontinuity at the origin so Fubini's Theorem does not apply.

Can you compute these by hand?(not required)

#6. The order of integration in a double integral is largely a matter of choice but sometimes the order can be the difference between a straightforward evaluation as opposed to a very difficult if not impossible evaluation.

Consider the function $f(x, y) = x \cos(xy)$ on $R = \left[0, \frac{\pi}{2}\right] \times [0, 1]$.

A. Integrate the function over the specified region with respect to y first (i.e. dydx).

$$f(x, y) = x \cos(xy) \text{ on } R = \left[0, \frac{\pi}{2}\right] \times [0, 1].$$

$$\int_0^{\frac{\pi}{2}} \int_0^1 x \cos(xy) dy dx = \int_0^{\frac{\pi}{2}} \left(\sin(xy) \Big|_0^1 \right) dx = \int_0^{\frac{\pi}{2}} \sin(x) dx = -\cos(x) \Big|_0^{\frac{\pi}{2}} = 1$$

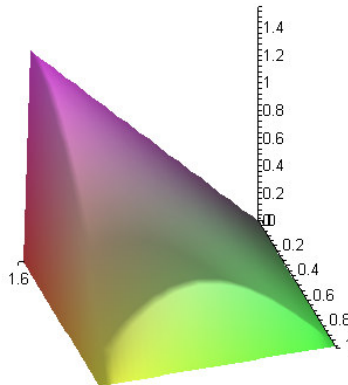
B. Integrate the function over the specified region with respect to x first and indicate when the difficulty arises. (i.e. dx dy).

$$f(x, y) = x \cos(xy) \text{ on } R = \left[0, \frac{\pi}{2}\right] \times [0, 1].$$

$$\int_0^1 \int_0^{\frac{\pi}{2}} x \cos(xy) dx dy = \int_0^1 \left(\frac{x \sin(xy)}{y} \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin(xy)}{y} dx \right) dy = \int_0^1 \left(\frac{x \sin(xy)}{y} \Big|_0^{\frac{\pi}{2}} + \frac{\cos(xy)}{y^2} \Big|_0^{\frac{\pi}{2}} \right) dy \rightarrow$$

$$= \int_0^1 \frac{\pi \sin\left(\frac{\pi y}{2}\right)}{y} dy + \int_0^1 \left(\frac{\cos\left(\frac{\pi y}{2}\right)}{y^2} - \frac{1}{y^2} \right) dy$$

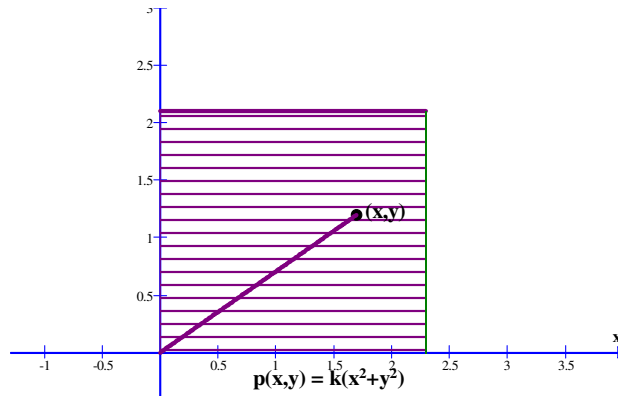
$$= \int_a^b \frac{\sin(y)}{y} dy / * \text{ is not an elementary integral.}$$



#7. A rectangular plate of sides lengths a and b is subjected to a normal force (that is perpendicular to the plate). The pressure, p , at any point on the plate is proportional to the square of the distance of that point from one corner. Find the total force on the plate [Note: Pressure is force per unit area].

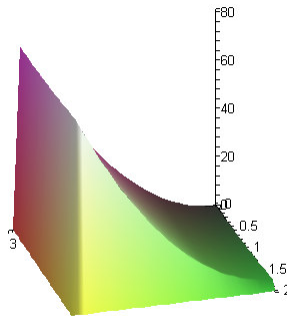
$$F = \int_0^a \int_0^b k(x^2 + y^2) dy dx = k \int_0^a \left(x^2 y + \frac{y^3}{3} \right) \Big|_0^b dx = k \int_0^a \left(x^2 b + \frac{b^3}{3} \right) dx = k \left(\frac{x^3 b}{3} + \frac{b^3 x}{3} \right) \Big|_0^a \rightarrow$$

$$= \frac{abk}{3} (a^2 + b^2)$$



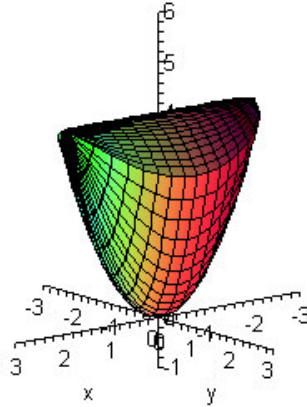
#8. Find the volume of the solid bounded by the paraboloid $z = 9x^2 + y^2$ above, by the plane $z = 0$ below, and laterally by the planes $x = 0$, $y = 0$, $x = 3$, and $y = 2$. Sketch the Region in the xy – plane and indicate your directions of integrations

$$V = \int_0^2 \int_0^3 (9x^2 + y^2) dx dy = \int_0^2 (3x^3 + y^2 x) \Big|_0^3 dy = \int_0^2 (81 + 3y^2) dy = 170$$



#9. Find the volume of the solid bounded by the two surfaces. Sketch the Region in the xy – plane and indicate your directions of integrations.

$$z = x^2 + 3y^2 \quad \& \quad z = 4 - y^2.$$



$$f(x, y) = 4 - y^2 - (x^2 + 3y^2) = 4 - x^2 - 4y^2; x^2 + 3y^2 = 4 - y^2 \Leftrightarrow \frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$V = 4 \int_0^1 \int_0^{2\sqrt{1-y^2}} (4 - x^2 - 4y^2) dx dy = 4 \int_0^1 \left(4x - \frac{x^3}{3} - 4y^2x \right) \Big|_0^{2\sqrt{1-y^2}} dy$$

$$= 4 \int_0^1 \left(8\sqrt{1-y^2} - \frac{8(1-y^2)\sqrt{1-y^2}}{3} - 8y^2\sqrt{1-y^2} \right) dy$$

$$= \frac{32}{3} \int_0^1 \left(2\sqrt{1-y^2} - 2y^2\sqrt{1-y^2} \right) dy$$

$$= \frac{64}{3} \int_0^1 \left(\sqrt{1-y^2} - y^2\sqrt{1-y^2} \right) dy$$

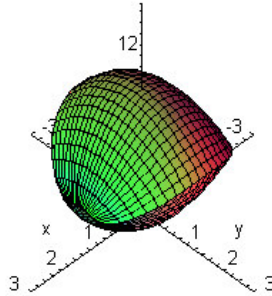
$$= \frac{64}{3} \int_0^{\frac{\pi}{2}} (\cos^2 \theta - \sin^2 \theta \cos^2 \theta) d\theta = \frac{64}{3} \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta \quad /* y = \sin \theta$$

$$*/ \int \cos^4 \theta d\theta = \frac{3\theta}{8} + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32} + C$$

$$\Rightarrow \frac{64}{3} \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta = \frac{64}{3} \left(\frac{3\theta}{8} + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32} \right) \Big|_0^{\frac{\pi}{2}}$$

$$= 4\pi$$

#10. Find the volume of the solid formed by the two paraboloids $z = x^2 + 3y^2$ & $z = 9 - 2x^2 - y^2$. Sketch the Region in the xy – plane and indicate your directions of integrations.



$$f(x, y) = 9 - 3x^2 - 4y^2; \frac{x^2}{3} + \frac{y^2}{\frac{9}{4}} = 1$$

$$V = 4 \int_0^{\frac{3}{2}} \int_0^{\frac{1}{\sqrt{3}}\sqrt{9-4y^2}} (9 - 3x^2 - 4y^2) dx dy = 4 \int_0^{\frac{3}{2}} (9x - x^3 - 4y^2x) \Big|_0^{\frac{1}{\sqrt{3}}\sqrt{9-4y^2}} dy$$

$$= 4 \int_0^{\frac{3}{2}} \left(\frac{9}{\sqrt{3}} \sqrt{9-4y^2} - \frac{(9-4y^2)\sqrt{9-4y^2}}{3\sqrt{3}} - \frac{4}{\sqrt{3}} y^2 \sqrt{9-4y^2} \right) dy$$

$$= \frac{4}{3\sqrt{3}} \int_0^{\frac{3}{2}} (18\sqrt{9-4y^2} - 8y^2\sqrt{9-4y^2}) dy \quad /* \text{let } u = 2y$$

$$= \frac{4}{3\sqrt{3}} \int_0^3 (9\sqrt{9-u^2} - u^2\sqrt{9-u^2}) dy \quad /* \text{let } u = 3\sin\theta$$

$$= \frac{4}{3\sqrt{3}} \int_0^{\frac{\pi}{2}} 81(\cos^2\theta - \sin^2\theta\cos^2\theta) d\theta = \frac{108}{\sqrt{3}} \int_0^{\frac{\pi}{2}} \cos^4\theta d\theta$$

$$*/ \int \cos^4\theta d\theta = \frac{3\theta}{8} + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32} + C$$

$$\Rightarrow \frac{108}{\sqrt{3}} \int_0^{\frac{\pi}{2}} \cos^4\theta d\theta = \frac{108}{\sqrt{3}} \left(\frac{3\theta}{8} + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32} \right) \Big|_0^{\frac{\pi}{2}} = \frac{27\sqrt{3}\pi}{4}$$