Vector calculus in an oblique basis set

by Kevin Gibson

Introduction

In almost all mathematical treatments of vector calculus an assumption is made that the basis set is orthogonal in nature. In general this may not always be the most convenient way to work a problem. This paper addresses the topic of performing vector analysis for any coordinate system without prior orthogonalization . In these papers Einstein's summation convention as well as covariant & contravariant notation will not be used. Moreover the metric for a given geometry will be expressed in terms of *scale factors*.

$$h_{\alpha\beta}^2 \equiv g_{\alpha\beta} \tag{1}$$

In this manner the general Pythagorean theorem would have the form

$$\mathrm{ds}^2 = \sum_{\alpha,\beta} \mathrm{h}_{\alpha\beta}^2 \cdot \mathrm{dx}_\alpha \cdot \mathrm{dx}_\beta \tag{2}$$

For example the scale factors for the cylindrical polar coordinates (r,θ,z) would be (1,r,1) and the Pythagorean theorem would have the form.

$$ds^2 = dr^2 + (r \cdot d\,\theta)^2 + dz^2 \tag{3}$$

The dot product

Usually the dot product between to vectors **A** and **B** is expressed as

$$\vec{A} \cdot \vec{B} = \sum_{\alpha} A_\alpha \cdot B_\alpha \tag{4}$$

However this assumes an orthogonal basis, in which like basis vectors multiply to yield 1 and unlike basis vectors multiply to 0. In this case all cross terms drop out. But what if one has an oblique, or non-orthogonal, basis? To answer this question, consider the vector defined below

$$\vec{ds} = \sum_{\alpha} h_\alpha \cdot dx_\alpha \cdot \hat{x}_\alpha \tag{5}$$

Using the shorthand notation that h_α is the same as $h_{\alpha\alpha}$. The dot product of this vector with itself (assuming an orthogonal basis) is

$$ds^2 = \sum_{\alpha} h_\alpha^2 \cdot dx_\alpha^2 = \sum_{\alpha} g_\alpha \cdot dx_\alpha^2 \tag{6}$$

Which is of course is the invariant infinitesimal distance or interval in space ds^2. With an oblique basis we want the same meaning, but now we need the general equation

$$ds^2 = \sum_{\alpha,\beta} g_{\alpha\beta}\, dx_\alpha\, dx_\beta = \sum_{\alpha,\beta} h_{\alpha\beta}^2\, dx_\alpha\, dx_\beta \tag{7}$$

As a proposed generalization of (4), assume that the general dot product has the form

$$\vec{A}\cdot\vec{B} = \sum_{\alpha,\beta} M_{\alpha\beta}\cdot(A_\alpha B_\beta) \tag{8}$$

$M_{\alpha\beta}$ is to be determined. Doing the dot product of d**s** with itself ala (8) produces

$$ds^2 = \sum_{\alpha,\beta} M_{\alpha\beta}(h_\alpha h_\beta)(dx_\alpha\, dx_\beta) \tag{9}$$

Making this equal the distance squared ds^2 in (2) means $M_{\alpha\beta}$ must be

$$M_{\alpha\beta} = \frac{h_{\alpha\beta}^2}{h_\alpha h_\beta} \tag{10}$$

The general dot product therefore is

$$\vec{A}\cdot\vec{B} = \sum_{\alpha,\beta} \frac{h_{\alpha\beta}^2}{h_\alpha h_\beta}\cdot(A_\alpha B_\beta) \tag{11}$$

It will be observed that in the case of a completely orthogonal basis this reverts back to (4).

Vector calculus

Having expanded the dot product to cover oblique coordinate systems, the next task is to apply the results to vector calculus. For reasons to be discussed later we will use the symbol □ to denote a more generalized del operator.

General gradient

Since a dot product is not involved in gradients, the gradient for a non-orthogonal space will then be the same as for any curvilinear space.

$$\vec{\square}\phi = \sum_{\alpha} \frac{\hat{x}_\alpha}{h_\alpha}\cdot\partial_\alpha\phi \tag{12}$$

Where the ∂_α signifies the partial derivative with respect to x_α.

General divergence
The divergence in an orthogonal 3-D curvilinear coordinate system can be expressed as follows[1]

$$\vec{\nabla}\cdot\vec{v}=\frac{1}{\Pi}\sum_j \partial_j\left(\frac{\Pi\cdot v_j}{h_j}\right) \tag{13}$$

With Π the product of all h_α. This can also be expressed as the dot product between the vector **v** and a del operator with components

$$\nabla_j^{op}\equiv\frac{1}{\Pi}\cdot\partial_j\frac{\Pi}{h_j} \tag{14}$$

Expanding the dot product to allow for non-orthogonal coordinate systems using the above gives

$$\vec{\Box}\cdot\vec{v}=\sum_{\alpha,\beta}\frac{h_{\alpha\beta}^2}{\Pi\, h_\alpha h_\beta}\cdot\partial_\alpha\left(\frac{\Pi\, v_\beta}{h_\alpha}\right) \tag{15}$$

Laplacian
The Laplacian is defined as the divergence of the gradient of some function ϕ.

$$\Box^2\phi=\sum_{\alpha,\beta}\frac{h_{\alpha\beta}^2}{\Pi\, h_\alpha h_\beta}\cdot\partial_\alpha\left[\frac{\Pi}{h_\alpha}\cdot(\vec{\Box}\phi)_\beta\right] \tag{16}$$

Or after simplifying,

$$\Box^2\phi=\sum_{\alpha,\beta}\frac{h_{\alpha\beta}^2}{\Pi\, h_\alpha h_\beta}\cdot\partial_\alpha\left(\frac{\Pi}{h_\alpha h_\beta}\cdot\partial_\beta\phi\right) \tag{17}$$

Generalizing to N space
Expanding (12), (15) and (17) into any N-space is trivial. All that is needed is to expand the summation over additional coordinates. The reason for the choice of symbol for del can be seen by applying this to the case of the Minkowski space where the Lorentz transformation holds. In this space let the coordinates be (x,y,z,t) and h = (1,1,1,ic). Applying the generalized

definition of the Laplacian gives

$$\Box_L^2 \phi = \partial_x^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi - \frac{1}{c^2} \partial_t^2 \phi \tag{18}$$

Which is the d' Alembertian operator for Cartesian coordinates[2] operating on ϕ. However the Laplacian above has been derived for any N dimensional geometry.

Works Cited

1. George Arfken and Hans Weber *Mathematical Methods for Physicists* 4th ed. p. 106
2. David Griffiths *Introduction to Electrodynamics* 2nd ed. p. 318

Contact information
kevin.gibson@asu.edu
kevinlg@mesacc.edu